MATH 7B – Sample Final Solutions

This test is in two parts. On part one, you may not use a calculator; on part two, a calculator is necessary. When you complete part one, you turn it in and get part two. Once you have turned in part one, you may not go back to it.

PART ONE - NO CALCULATORS ALLOWED

(1) Find each of the following:
(Note: answers to inverse trig. problems should be in radians, not degrees)

(a) $\sin^{-1}(-1) = -\frac{\pi}{2}$

(b) $\tan^{-1}(0) = 0$

(c) $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$

(d) $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

(e) $\tan 330° = -\frac{1}{\sqrt{3}}$

(f) $\cos^{-1}\left(\frac{-\sqrt{2}}{2}\right) = \frac{3\pi}{4}$

(g) $\sec\left(\frac{5\pi}{6}\right) = -\frac{2}{\sqrt{3}}$

(h) $\csc(\pi) =$ undefined

i) $\cos^{-1}\left(\cos\left(\frac{3\pi}{2}\right)\right) = \frac{\pi}{2}$

(j) $\tan\left(\tan^{-1}(1/3)\right) = \frac{1}{3}$

(2) Fill in the blank to complete the identity.

(a) $\sin 2\theta = 2\sin\theta\cos\theta$

(b) $\cos^2 x = 1-\sin^2 x$ or $\frac{1+\cos 2x}{2}$

(c) $\sin(\theta/2) = \pm\sqrt{\frac{1-\cos\theta}{2}}$

(d) $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

MATH 7B – Sample Final Exam – Part Two

Solutions

Fill in the blanks. In problems 1 - 7 fill in the blank with the most appropriate answer

(1) $\sin(-\theta) =$ $-\sin\theta$

(2) The graph of the polar curve $r=4\sin\theta$ is a circle center $(0,2)$ rect, radius $= 2$

(3) $\begin{vmatrix} -5 & 3 \\ 2 & -7 \end{vmatrix} =$ 29

(4) The period of $f(x) = \tan(3\pi x)$ is $\frac{\pi}{3\pi} = \frac{1}{3}$

(5) The range of $f(x) = \cos^{-1}x$ is $[0, \pi]$

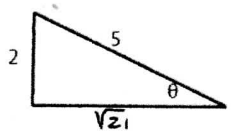
(6) The graph of $x^2 + 5y^2 + 4x + 10y - 2 = 0$ is a/an ellipse

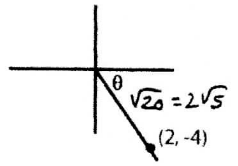
(7) The range of $f(x) = \tan x$ is $(-\infty, \infty)$

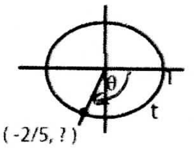
(8) Convert the polar point $(7, 11\pi/6)$ to rectangular coordinates $\left(\frac{7\sqrt{3}}{2}, -\frac{7}{2}\right)$

$x = r\cos\theta = 7\cos\frac{11\pi}{6}$ $\quad y = r\sin\theta = 7\sin\frac{11\pi}{6}$

(9) Given the following figures, find:







$\frac{4}{25} + y^2 = 1$

$y^2 = \frac{21}{25}$

$y = -\frac{\sqrt{21}}{5}$

(a) $\cos\theta =$ $\frac{\sqrt{21}}{5}$

(b) $\theta \approx$ $\cos^{-1}\frac{\sqrt{21}}{5}$ or $\sin^{-1}\frac{2}{5}$ degrees

(c) $\sin\theta =$ $\frac{-4}{2\sqrt{5}} = \frac{-2}{\sqrt{5}}$

(d) $\theta \approx$ $\tan^{-1}\left(\frac{-4}{2}\right)$ degrees

(e) $\sin t =$ $-\frac{\sqrt{21}}{5}$

(f) $\theta \approx$ $\sin^{-1}\left(+\frac{\sqrt{21}}{5}\right) + 180^\circ$ degrees

A little tricky, θ is a negative angle in figure

(10) Given the point $(-4, -4)$ in rectangular coordinates, find two different polar representations; one with $r > 0$, the other with $r < 0$.

point in Q III

$r^2 = 16 + 16 = 32$ $\quad R = \pm 4\sqrt{2}$

$\tan\theta = \frac{-4}{-4} = 1$

$(4\sqrt{2}, 225^\circ)$ $(-4\sqrt{2}, 45^\circ)$

(11) Given the following matrices:

$A = \begin{bmatrix} 2 & -1 \\ 3 & -5 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 1 & 5 \\ 0 & 4 & 3 \\ 1 & -2 & 3 \end{bmatrix}$ $C = \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix}$ Find the following, if possible. (If not possible, say so.)

(a) A^{-1}

formula, or

$\left[\begin{array}{cc|cc} 2 & -1 & 1 & 0 \\ 3 & -5 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & -1/2 & 1/2 & 0 \\ 3 & -5 & 0 & 1 \end{array}\right]$

$-3R_1 + R_2$ $\left[\begin{array}{cc|cc} 1 & -1/2 & 1/2 & 0 \\ 0 & -7/2 & -\frac{3}{2} & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & -1/2 & 1/2 & 0 \\ 0 & 1 & \frac{3}{7} & \frac{-2}{7} \end{array}\right]$

$\frac{1}{2}R_2 + R_1$ $\left[\begin{array}{cc|cc} 1 & 0 & 5/7 & -1/7 \\ 0 & 1 & 3/7 & -2/7 \end{array}\right]$ $B^{-1} = \begin{bmatrix} 5/7 & -1/7 \\ 3/7 & -2/7 \end{bmatrix}$

(b) AC

$\begin{bmatrix} -1 & -13 \\ -12 & -44 \end{bmatrix}$

(e) det(B)

$3(18) + 1(-17) = 37$

(12) SOLVE the following equations: $0 \le x < 2\pi$

(a) $\sin 2x = 3\sin x$

$2\sin x\cos x - 3\sin x = 0$

$\sin x(2\cos x - 3) = 0$

$\sin x = 0 \quad \cos x = \frac{3}{2}$

never

$\boxed{x = 0, \pi}$

(b) $\cos^2(3x) - 1 = 0$

$\cos^2(3x) = 1$

$\cos(3x) = \pm 1$

$3x = \pi k$

$x = \frac{\pi}{3} k$

in $[0, 2\pi)$

$\boxed{x = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}}$

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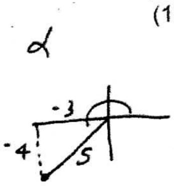
(13) Given $\csc\alpha = -5/4$, α in the third quadrant, and $\beta = \sin^{-1}(2/3)$, (12 points)

Find:

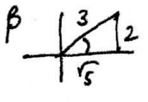


a) $\sin\left(\frac{\alpha}{2}\right)$ Note $\pi < \alpha < \frac{3\pi}{2}$

$\Rightarrow \frac{\pi}{2} < \frac{\alpha}{2} < \frac{3\pi}{4}$

So $\frac{\alpha}{2}$ in QII

$\sin\frac{\alpha}{2} = +\sqrt{\frac{1-\cos\alpha}{2}} = \sqrt{\frac{1+3/5}{2}} = \sqrt{\frac{8}{10}} = \frac{2}{\sqrt{5}}$



b) $\tan 2\beta = \frac{2\tan\beta}{1-\tan^2\beta} = \frac{2\cdot\frac{2}{\sqrt{5}}}{1-\frac{4}{5}} = \frac{\frac{4}{\sqrt{5}}}{\frac{1}{5}} = \frac{20}{\sqrt{5}} = 4\sqrt{5}$

c) $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$= \frac{-3}{5}\frac{\sqrt{5}}{3} - \frac{-4}{5}\frac{2}{3} = \frac{8-3\sqrt{5}}{15}$

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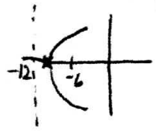
(14) Verify the identity: $\frac{1-\sin\theta}{\cos\theta} + \frac{\cos\theta}{1-\sin\theta} = 2\sec\theta$ (10 points)

LHS $\frac{1-\sin\theta}{\cos\theta} + \frac{\cos\theta}{1-\sin\theta} = \frac{(1-\sin\theta)^2 + \cos^2\theta}{\cos\theta(1-\sin\theta)} = \frac{1-2\sin\theta+\overbrace{\sin^2\theta+\cos^2\theta}^{1}}{\cos\theta(1-\sin\theta)}$

$= \frac{2-2\sin\theta}{\cos\theta(1-\sin\theta)} = \frac{2(1-\sin\theta)}{\cos\theta(1-\sin\theta)} = \frac{2}{\cos\theta} = 2\sec\theta. = RHS$

So LHS = RHS

(15) Find an equation of the parabola having focus (-6,0) and directrix x = -12.



vertex must be equidistant from both, so vertex is $(-9,0)$

and $p=3$ so

$$y^2 = 4px$$

$$y^2 = 12(x+9)$$

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(16) Use Gaussian Elimination OR Cramer's Rule to solve:
(no credit if requested method is not used)

$$\begin{cases} 3x-y-z=8 \\ x+y-2z=5 \\ 2x-y+z=1 \end{cases} \quad \begin{bmatrix} 3 & -1 & -1 & 8 \\ 1 & 1 & -2 & 5 \\ 2 & -1 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 1 & -2 & 5 \\ 3 & -1 & -1 & 8 \\ 2 & -1 & 1 & 1 \end{bmatrix} \xrightarrow[-2R_1+R_3 \to R_3]{-3R_1+R_2 \to R_2} \begin{bmatrix} 1 & 1 & -2 & 5 \\ 0 & -4 & 5 & -7 \\ 0 & -3 & 5 & -9 \end{bmatrix}$$

$$\begin{matrix} -R_3+R_2 \to R_2 \\ \text{then } -R_2 \to R_2 \end{matrix} \begin{bmatrix} 1 & 1 & -2 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & -3 & 5 & -9 \end{bmatrix} \xrightarrow[3R_2+R_3 \to R_3]{} \begin{bmatrix} 1 & 1 & -2 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 5 & -15 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & -2 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

$$2R_3+R_1 \to R_1 \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -3 \end{bmatrix} \qquad (1,-2,-3)$$

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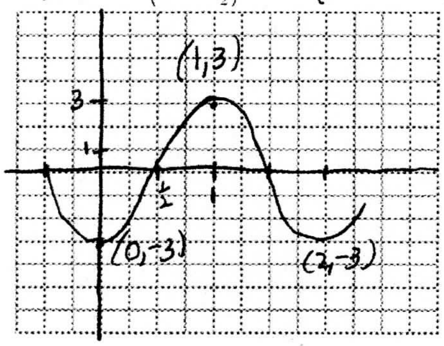
(17) Graph the following function. Show work.

$f(x) = 3\sin\left(\pi x - \frac{\pi}{2}\right) = 3\sin\left(\pi\left(x-\frac{1}{2}\right)\right)$ (one period, label highs and lows, show scale)

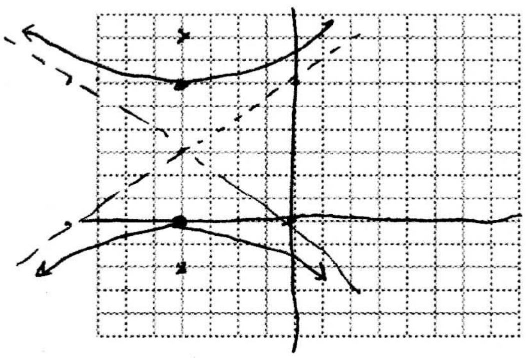
period $= \frac{2\pi}{\pi} = 2$

shift right $\frac{1}{2}$



(18) Carefully sketch the graph of $9x^2 - 16y^2 + 72x + 96y + 144 = 0$, and find the following desired information. Label at least 2 points on your graph and show scale. (10 points)

VERTICES: $(-4,0)$ $(-4,6)$ FOCI: $(-4,8)(-4,-2)$



$9(x^2+8x \quad) - 16(y^2-6y \quad) = -144$

$9(x^2+8x+16) - 16(y^2-6y+9) = -144+144-144$

$9(x+4)^2 - 16(y-3)^2 = -144$

$\frac{(y-3)^2}{9} - \frac{(x+4)^2}{16} = 1$

$c^2 = a^2+b^2 = 9+16 = 25$

$c = 5$

(19) Given the vectors **W** = <-4,-3> and **V** = < 2, 5>, find the following:

a) || **W** || $= \sqrt{16+9}$ 5

b) **W** • **V** -23

c) Find the direction angle of **w** (exactly) $\tan^{-1}(\frac{3}{4}) + 180°$

d) The direction angle of **v** (exactly) $\tan^{-1}(\frac{5}{2})$

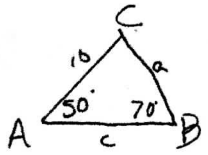
e) Find b so that <b,7> is orthogonal to **W** $b = \frac{-21}{4}$

$-4b - 21 = 0$

f) Find the angle between **w** and **v** $\theta = \cos^{-1}\left(\frac{-23}{5\sqrt{29}}\right)$

$\cos\theta = \frac{\vec{w} \cdot \vec{v}}{\|\vec{w}\| \|\vec{v}\|} = \frac{-23}{5\sqrt{29}}$

(20) Given triangle ABC with A=50°, B=70° and b=10 inches, find the remaining parts.



$C = 60°$

$\frac{a}{\sin 50°} = \frac{c}{\sin 60°} = \frac{10}{\sin 70°}$

$a = \frac{10\sin 50°}{\sin 70°} \approx 8.15''$

$c = \frac{10\sin 60°}{\sin 70°} \approx 9.22''$

Find all solutions to the following equations.

(21) $3\tan^2x - \sec^2x - 5 = 0$

$3(\sec^2x - 1) - \sec^2x - 5 = 0$

$2\sec^2x - 8 = 0$

$\sec^2x = 4$

$\sec x = \pm 2$

$\cos x = \pm 1/2$

$x = \frac{\pi}{3} + \pi k, \ \frac{2\pi}{3} + \pi k$

(22) $\cos(2x) = 2 + 5\cos x$

$2\cos^2x - 1 = 2 + 5\cos x$

$2\cos^2x - 5\cos x - 3 = 0$

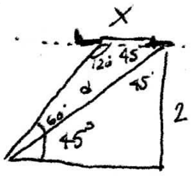
$(2\cos x + 1)(\cos x - 3) = 0$

$\cos x = -1/2, \quad \cos x = 3$

$x = \frac{2\pi}{3} + 2\pi k$

$\frac{4\pi}{3} + 2\pi k$

(23) A man looks up and sees an airplane flying in his direction at a level altitude of 2 miles. He watches the airplane for a few minutes. During that period of time he notices that the angle of elevation to the airplane changes from 45° to 60°. How far has the plane traveled in that time? many approaches... can do using two right triangles:



Right triangle

$\frac{2}{d} = \sin 45° = \frac{1}{\sqrt{2}}$

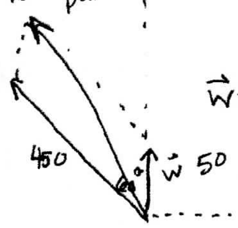
$d = 2\sqrt{2}$

obtuse angle.. - Law of sines

$\frac{d}{\sin 120°} = \frac{x}{\sin 15°}$

$x = \frac{d\sin 15°}{\sin 120°} = \frac{2\sqrt{2}\sin 15°}{\sqrt{3}/2} = \frac{4\sqrt{2}\sin 15°}{\sqrt{3}} \approx 0.85$ miles

(24) An airplane is traveling at a constant airspeed of 450 mph in the direction N60°W. If wind is blowing directly northward at a rate of 50 mph, what is the actual speed and direction of the airplane relative to the ground?



$\vec{W} = \langle 0, 50 \rangle$

$\vec{R} = \vec{p} + \vec{w} = \langle -225\sqrt{3}, 275 \rangle$

$\|\vec{R}\| = \sqrt{(-225\sqrt{3})^2 + (275)^2} = 477$ mph actual speed

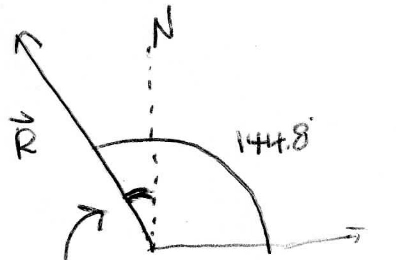
$\tan\theta = \frac{275}{-225\sqrt{3}}$, θ in Quad II

so $\theta = 180° - \tan^{-1}\left(\frac{275}{225\sqrt{3}}\right) \approx 144.8°$

$\vec{P} = \langle 450\cos 150°, 450\sin 150° \rangle$

$= \langle 450 \cdot -\frac{\sqrt{3}}{2}, 450 \cdot \frac{1}{2} \rangle$

$= \langle -225\sqrt{3}, 225 \rangle$



direction is N 54.8° W