MATH 5B – SAMPLE FINAL EXAM

SHOW ALL WORK NEATLY AND CLEARLY BOX ALL ANSWERS.

FILL IN THE BLANK WITH THE MOST APPROPRIATE ANSWER. NO PARTIAL CREDIT.

(1) TRUE OR FALSE: If $\lim_{k\to\infty} a_k = 0$ then $\sum_{k=1}^{\infty} a_k$ converges. false

(2) $\int \cos^2 x \, dx =$ $\frac{1}{2}x+\frac{1}{4}\sin 2x+C$

(3) cosh(ln 3) = $\frac{e^{\ln 3}+e^{-\ln 3}}{2} = \frac{3+\frac{1}{3}}{2}$ (exactly) $\frac{10}{6} = \frac{5}{3}$

(4) Express the point $(-\sqrt{3}, 1)$ in polar coordinates(exactly) $\left(2, \frac{5\pi}{6}\right)$ $r^2=4$ $r=\pm 2$ $\tan\theta = -\frac{1}{\sqrt{3}}$

(5) Express the polar point $(8, 7\pi/6)$ in rectangular coordinates (exactly) $(-4\sqrt{3}, -4)$

(6) The derivative of $f(x) = e^{1-3x}$ is $-3e^{1-3x}$

(7) $6 - 2 + 2/3 - 2/9 + \ldots\ldots = \frac{a}{1-r} = \frac{6}{1+\frac{1}{3}} = 9/$ geometric series $a=6$ $r=\left(\frac{-1}{3}\right)$

(8) $\lim_{x\to 0^+} \frac{\sin x}{x^2} =$ ∞ $\lim_{x\to 0^+} \frac{\sin x}{x^2} = \lim_{x\to 0^+} \frac{\cos x}{2x} = \infty$ $\frac{0}{0}$ $\frac{1}{0}$

(9) $\int \frac{1}{1-x} dx =$ $-\ln(1-x)+C$

(10) True or False: $1 - \frac{1}{\sqrt[4]{2}} + \frac{1}{\sqrt[4]{3}} - \frac{1}{\sqrt[4]{4}} + \ldots$ is a conditionally convergent series. true

(11) For each of the following series, classify as convergent (absolute or conditional if applicable) or divergent. SHOW ALL DETAILS.

(a) $\frac{1}{e} + \frac{2}{e^4} + \frac{3}{e^9} + \frac{4}{e^{16}} + \ldots\ldots$

$$\sum_{n=1}^{\infty} \frac{n}{e^{n^2}} = \sum_{n=1}^{\infty} ne^{-n^2}$$

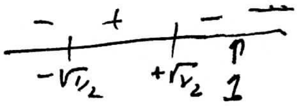
Consider

$f(x) = xe^{-x^2}$

This function is cont's.

$f'(x) = xe^{-x^2}(-2x) + e^{-x^2}$

$= (-2x^2+1)e^{-x^2}$



so f is decreasing for $x>1$

$\lim_{x\to\infty} xe^{-x^2} = \lim_{x\to\infty} \frac{x}{e^{x^2}} = \lim_{x\to\infty} \frac{1}{2xe^{x^2}} = 0$

So... integral test applies

$\int_0^{\infty} xe^{-x^2} dx = \lim_{b\to\infty} \int_0^b xe^{-x^2} dx = \lim_{b\to\infty} -\frac{1}{2}e^{-b^2} + \frac{1}{2}$

$= \frac{1}{2}$

now $\int xe^{-x^2} dx = -\frac{1}{2}\int e^u du = -\frac{1}{2}e^{-x^2}+C$

$u = -x^2$

$du = -2x\,dx$

so since integral converges, series does (absolute)

(b) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2n}{4n^2-3}$

Check for absolute convergence

Limit comparison test

$$\lim_{n\to\infty} \frac{\frac{2n}{4n^2-3}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{2n^2}{4n^2-3} = \frac{1}{2}$$

so since $\sum \frac{1}{n}$ diverges,

$\sum \frac{2n}{4n^2-3}$ diverges, so given

series is not absolutely convergent.

Conditionally ?

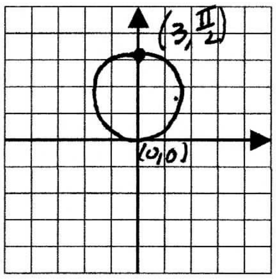
$\lim_{n\to\infty} \frac{2n}{4n^2-3} = 0$ ✓

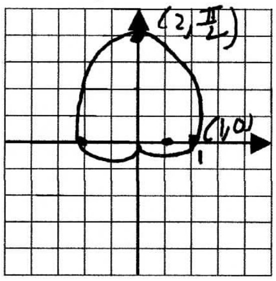
$f(x) = \frac{2x}{4x^2-3} \Rightarrow f'(x) = \frac{(4x^2-3)2 - 2x(8x)}{(4x^2-3)^2}$

So decreasing $= -\frac{6+8x^2}{(4x^2-3)^2} < 0 \ \forall x$

So Conditionally convergent

(12) Sketch the graphs of the polar curves: r = 3sinθ. and r = 1+sinθ.





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(13) Find each of the following limits. Show details or no credit will be given:

(a) $\lim_{x\to0}\left(\frac{1-\cos x}{x^2}\right)$ $\left(\frac{0}{0}\right)$

L'Hop

$= \lim_{x\to0}\frac{\sin x}{2x}$ $\left(\frac{0}{0}\right)$

L'Hop

$= \lim_{x\to0}\frac{\cos x}{2}$

$= \frac{1}{2}$

(b) $\lim_{x\to\infty}(x\tan(1/x)) = \lim_{x\to\infty}\frac{\tan(1/x)}{1/x}$ $\frac{0}{0}$

$\infty\cdot0$

L'Hop

$= \lim_{x\to\infty}\frac{\left(\frac{-1}{x^2}\right)\sec^2\left(\frac{1}{x}\right)}{\left(\frac{-1}{x^2}\right)}$

$= \lim_{x\to\infty}\sec^2\left(\frac{1}{x}\right)$

$= \sec^2 0 = 1$

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(14) If f(x) = 2 + e^{-x}, find f^{-1} (x).

$y = 2 + e^{-x}$

switch

$x = 2 + e^{-y}$

$x - 2 = e^{-y}$

$\ln(x-2) = -y$

$y = -\ln(x-2)$

$f^{-1}(x) = -\ln(x-2)$

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(15) Determine the interval of convergence: SHOW DETAILS.

$$\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{3^n(n+1)}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{3^{n+1}(n+2)}\cdot\frac{3^n(n+1)}{x^n}\right| = \frac{|x|}{3}\cdot\frac{n+1}{n+2}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x|}{3}\frac{n+1}{n+2} = \frac{|x|}{3}.$$

So series converges absolutely when $\frac{|x|}{3} < 1$ and diverges when $\frac{|x|}{3} > 1$

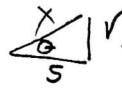
Consider $\frac{|x|}{3} = 1 \Rightarrow |x| = 3$.

$x = 3$ $\sum\frac{(-1)^n}{n+1}$ converges by AST

$x = -3$ $\sum\frac{1}{n+1}$ diverges by comparison to harmonic series

$\Longrightarrow$ $(-3, 3]$

(16) Compute each of the following integrals. For any given <u>improper</u> integrals, you must first write the integral in terms of the limit as it is defined.



(a) $\int \frac{2x^2-x+4}{x^3+4x}\,dx$ ~~by~~ partial Fraction

$$\frac{2x^2-x+4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4} = \frac{1}{x} + \frac{x-1}{x^2+4}$$

$$\int\left(\frac{1}{x} + \frac{x-1}{x^2+4}\right)dx$$

$$= \int \frac{1}{x}\,dx + \int \frac{x}{x^2+4}\,dx - \int \frac{1}{x^2+4}\,dx$$

$u = x^2+4$
$du = 2x\,dx$

$$\ln|x| + \frac{1}{2}\ln(x^2+4) - \frac{1}{2}\tan^{-1}\frac{x}{2} + C$$

(8) $\int \frac{dx}{x^2\sqrt{x^2-25}}$ $x = 5\sec\theta$
$dx = 5\sec\theta\tan\theta\,d\theta$

$$\int \frac{5\sec\theta\tan\theta\,d\theta}{25\sec^2\theta\sqrt{25\sec^2\theta-25}}$$

$$\int \frac{5\sec\theta\tan\theta}{25\sec^2\theta\;5\tan\theta}\,d\theta$$

$$= \int \frac{1}{25}\cos\theta\,d\theta$$

$$= \frac{1}{25}\sin\theta + C$$

$$= \frac{1}{25}\frac{\sqrt{x^2-25}}{x} + C$$

(c) $\int_0^\infty \frac{e^{-x}}{1+e^{-2x}}\,dx$

$u = e^{-x}$
$du = -e^{-x}$

$$\int_0^\infty \frac{e^{-x}}{1+e^{-2x}}dx = \lim_{b\to\infty}\int_0^b \frac{e^{-x}}{1+e^{-2x}}\,dx$$

$$= \lim_{b\to\infty}\int_{x=0}^{x=b} \frac{-du}{1+u^2} = \lim_{b\to\infty} -\tan^{-1}u\Big]_{x=0}^{x=b}$$

=

=

$$= \lim_{b\to\infty}\left(-\tan^{-1}e^{-x}\right)\Big]_0^b$$

$$= \lim_{b\to\infty}\left(-\tan^{-1}e^{-b} + \tan^{-1}1\right)$$

$$= \frac{\pi}{4}$$

(d) $\int \cos^3 x\,dx = \int \cos x \cos^2 x\,dx$

$$= \int \cos x\,(1-\sin^2 x)\,dx$$

$u = \sin x$
$du = \cos x\,dx$

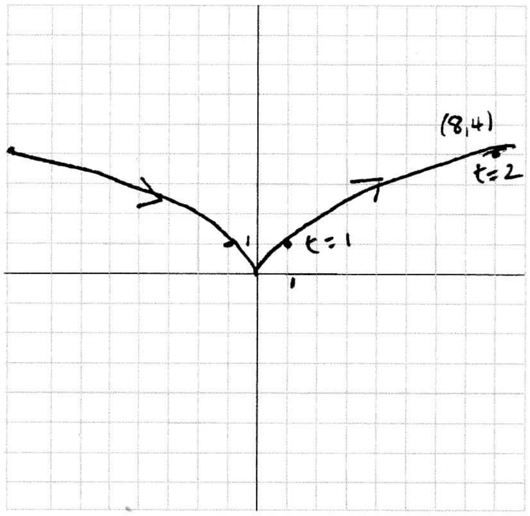
$$= \int (1-u^2)\,du$$

$$u - \frac{1}{3}u^3 + C$$

$$\sin x - \frac{1}{3}\sin^3 x + C$$

(17) (a) Sketch the curve given by the parametric equations $\begin{cases} x = t^3 \\ y = t^2 \end{cases}$ $\Rightarrow$ $y = x^{2/3}$

(b) Find the length of the portion of the above curve corresponding to $0 \le t \le 2$



$$L = \int_0^2 \sqrt{(3t^2)^2 + (2t)^2}\, dt$$

$$= \int_0^2 \sqrt{9t^4 + 4t^2}\, dt$$

$$= \int_0^2 t\sqrt{9t^2+4}\, dt \qquad u = 9t^2+4 \quad du = 18t\,dt$$

$$= \frac{1}{18}\int_4^{40} u^{1/2}\, du = \frac{1}{18}\frac{2}{3}\, u^{3/2}\Big]_4^{40}$$

$$= \frac{1}{27}\left(40^{3/2} - 4^{3/2}\right) = \frac{1}{27}\left(80\sqrt{10} - 8\right) \approx 9.07$$

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(18) Given f(x) = xe^x , answer the following, find any other information necessary to obtain a graph and sketch the graph.

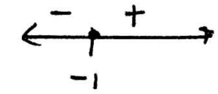
(a) $\lim_{x\to\infty} f(x)$ = ∞

(b) $\lim_{x\to-\infty} f(x)$ = 0

$$\lim_{x\to-\infty} xe^x = \lim_{x\to-\infty} \frac{x}{e^{-x}} = \lim_{x\to-\infty} \frac{1}{-e^{-x}} = \lim_{x\to-\infty} -e^{x} = 0$$

(c) local extrema local min $(-1, \frac{-1}{e}) \approx (-1, -.36)$

$y' = xe^x + e^x = (x+1)e^x$

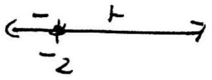


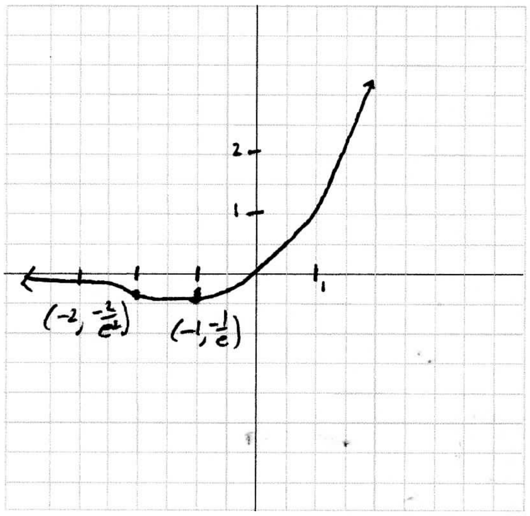
(d) Discuss concavity. CD$(-\infty, -2)$ CU$(2, \infty)$

inflection $(-2, \frac{-2}{e^2})$

$(-2, -.27)$

$y'' = xe^x + 2e^x = (x+2)e^x$





(19) (a) Use series to approximate $\int_0^{1/2} \frac{1}{\sqrt{1+x^2}}\,dx$ correct to 2 decimal places. error $< .005$

(b) Find the value of the integral exactly by integrating directly.

a) Find series for $(1+x)^{-1/2}$, then substitute x^2 for x and integrate.

$$f(x) = (1+x)^{-\frac{1}{2}}$$

$$f'(x) = -\tfrac{1}{2}(1+x)^{-3/2}$$

$$f''(x) = \tfrac{+1}{2}\tfrac{3}{2}(1+x)^{-\frac{5}{2}}$$

$$f'''(x) = \tfrac{-1}{2}\tfrac{3}{2}\tfrac{5}{2}(1+x)^{-\frac{7}{2}}$$

$$f^{(n)}(x) = \frac{(-1)^n\,1\cdot3\cdot5\cdots(2n-1)}{2^n}(1+x)^{-\frac{2n+1}{2}}$$

$$\text{so } f^{(n)}(0) = \frac{(-1)^n\,1\cdot3\cdots(2n-1)}{2^n} \text{ for } n \geq 1$$

$$\frac{1}{\sqrt{1+x}} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n\,1\cdot3\cdot5\cdots(2n-1)}{2^n\ n!}x^n$$

Substitute x2

$$\frac{1}{\sqrt{1+x^2}} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n\,1\cdot3\cdot5\cdots(2n-1)}{2^n n!}x^{2n}$$

Integrate.

$$\int_0^{1/2} \frac{1}{\sqrt{1+x^2}} = x + \sum_{n=1} (-1)^n \frac{(1\cdot3\cdots(2n-1))}{2^n n!}\frac{x^{2n+1}}{2n+1}\Bigg]_0^{1/2}$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n(1\cdot3\cdot5\cdots(2n-1))}{2^n n!\,(2n+1)}\left(\frac{1}{2}\right)^{2n+1}$$

$$= \frac{1}{2} - \frac{1}{2\cdot3\cdot2^3} + \frac{1\cdot3}{2^2\,2!\,5\,2^5} - \frac{1\cdot3\cdot5}{2^3\cdot3!\cdot7\cdot2^7} - \cdots$$

$$= \underbrace{\frac{1}{2} - \frac{1}{48}} + \underset{.002}{\frac{3}{1280}} + \cdots \approx 0.48$$

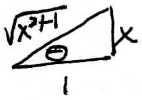
should agree to 2 dec. places

b) $\int_0^{1/2} \frac{1}{\sqrt{1+x^2}}\,dx$

Trig substitution

let $x = \tan\theta$

$dx = \sec^2\theta\, d\theta$



$$\int_0^{1/2} \frac{1}{\sqrt{1+\tan^2\theta}}\sec^2\theta\, d\theta$$

$$\int_{x=0}^{x=1/2} \sec\theta\, d\theta$$

$$= \ln|\sec\theta + \tan\theta|\Big]_{x=0}^{x=\frac{1}{2}}$$

$$= \ln|\sqrt{x^2+1} + x|\Big]_0^{1/2}$$

$$= \ln|\sqrt{\tfrac{5}{4}} + \tfrac{1}{2}| = \underset{\text{exact}}{\underline{\ln.\frac{\sqrt{5}+1}{2}}} \approx 0.4812.$$

or $\sinh^{-1}(\frac{1}{2})$