Mathematics 114Q
Integration Practice Problems

Name: SOLUTIONS

1. $\int (2x+5)(x^2+5x)^7 dx$

$$= \frac{1}{8}(x^2+5x)^8 + C$$

$$[u = x^2+5x]$$

2. $\int (3-x)^{10} dx$

$$= -\frac{1}{11}(3-x)^{11} + C$$

$$[u = 3-x]$$

3. $\int \sqrt{7x+9}\, dx$

$$= \frac{2}{21}(7x+9)^{3/2}$$

$$[u = 7x+9]$$

4. $\int \frac{x^3}{(1+x^4)^{1/3}} dx$

$$= \frac{3}{8}(1+x^4)^{2/3} + C$$

$$[u = 1+x^4]$$

5. $\int e^{5x+2} dx$

$$= \frac{1}{5}e^{5x+2} + C$$

$$[u = 5x+2]$$

6. $\int 4\cos(3x)dx$

$$= \frac{4}{3}\sin(3x) + C$$

$$[u = 3x]$$

7. $\int \frac{\sin(\ln x)}{x}dx$

$$= -\cos(\ln x) + C$$

$$[u = \ln x]$$

8. $\int \frac{x+2}{x^2+4x-3}dx$

$$= \frac{1}{2}\ln|x^2+4x-3| + C$$

$$[u = x^2+4x-3]$$

9. $\int \left(3^{x^2+1}\right)\left(x\right)\, dx$

$$= \frac{1}{2}\cdot\frac{1}{\ln 3}\cdot 3^{x^2+1} + C$$

$$[u = x^2+1]$$

10. $\int \frac{1}{x\ln x}dx$

$$= \ln|\ln x| + C$$

$$[u = \ln x]$$

11. $\int \frac{\cos(5x)}{e^{\sin(5x)}}dx$

$$= -\frac{1}{5}e^{-\sin(5x)} + C$$

$$[u = \sin(5x)]$$

12. $\int_0^{\sqrt{\pi}} x\sin(x^2)dx$

Let $u = x^2$. Then $du = 2x\,dx$, so

$$\begin{aligned}\int_0^{\sqrt{\pi}} x\sin(x^2)dx &= \frac{1}{2}\int_0^{\sqrt{\pi}} \sin(x^2)\cdot 2x\,dx \\ &= \frac{1}{2}\int_{x=0}^{x=\sqrt{\pi}} \sin(u)du \\ &= \frac{1}{2}\Big(-\cos(x^2)\Big)\Big|_0^{\sqrt{\pi}} \\ &= 1\end{aligned}$$

13. $\int x\sqrt{4-x}\,dx$

[Hint: If $u = 4 - x$, what does that make x in terms of u?]

If $u = 4-x$, then $x = 4-u$ and so $dx = -1du$. Now just substitute all of this into the integral:

$$\begin{aligned}\int x\sqrt{4-x}\,dx &= \int (4-u)\sqrt{u}\,(-1)du \\ &= \int \Big(-4u^{1/2} + u^{3/2}\Big)du \\ &= -\frac{8}{3}(4-x)^{3/2} + \frac{2}{5}(4-x)^{5/2} + C\end{aligned}$$

14. $\int xe^x dx$

$$= xe^x - e^x + C$$

$$[u = x, \quad dv = e^x dx]$$

15. $\int x \sin x \, dx$

$$= -x\cos(x) + \sin(x) + C$$

$$[u = x, \quad dv = \sin(x)dx]$$

16. $\int x \ln x \, dx$

$$= \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$$

$$[u = \ln x, \quad dv = xdx]$$

17. $\int \ln x \, dx$

$$= x\ln x - x + C$$

$$[u = \ln x, \quad dv = 1 \, dx]$$

18. $\int \frac{\ln x}{x^5} dx$

$$= -\frac{1}{4} \cdot \frac{\ln x}{x^4} - \frac{1}{16} \cdot \frac{1}{x^4} + C$$

$$[u = \ln x, \quad dv = x^{-5} dx]$$

19. $\int x^2 e^{3x} dx$

You will have to use integration by parts twice. First use $u = x^2$ and $dv = e^{3x}dx$. This will give you

$$\int x^2 e^{3x} dx = \frac{x^2}{3}e^{3x} - \frac{2}{3}\int xe^{3x}dx.$$

For the new integral, use $u = x$ and $dv = e^{3x}dx$. This gives the answer:

$$\begin{aligned}\int x^2 e^{3x} dx &= \frac{x^2}{3}e^{3x} - \frac{2}{3}\left[\frac{x}{3}e^{3x} - \frac{1}{3}\int e^{ex}dx\right] \\ &= \frac{x^2}{3}e^{3x} - \frac{2x}{9}e^{3x} + \frac{2}{27}e^{3x} + C\end{aligned}$$

20. $\int x^3 \ln(5x) dx$

$$= \frac{x^4}{4}\ln(5x) - \frac{x^4}{16} + C$$

$$[u = \ln(5x), \quad dv = x^3 dx]$$

21. $\int x\sqrt{x+3}\, dx$

Let $u = x$ and $dv = (x+3)^{1/2}dx$. Then

$$\begin{aligned}\int x\sqrt{x+3}dx &= \frac{2}{3}x(x+3)^{3/2} - \int \frac{2}{3}(x+3)^{3/2}dx \\ &= \frac{2}{3}x(x+3)^{3/2} - \frac{4}{15}(x+3)^{5/2} + C.\end{aligned}$$

22. $\int \sin^2(x)dx$

[Hint: write $\sin^2(x)$ as $\sin(x)\sin(x)$ and use the Pythagorean Theorem.]

As instructed in the hint, write $\sin^2(x) = \sin(x)\sin(x)$. Let $u = \sin(x)$ and $dv = \sin(x)dx$. Then

$$\begin{aligned}\int \sin^2(x)dx &= -\sin(x)\cos(x) + \int \cos^2(x)dx \\ &= -\sin(x)\cos(x) + \int \Big(1 - \sin^2(x)\Big)dx \\ &= -\sin(x)\cos(x) + x - \int \sin^2(x)dx\end{aligned}$$

Now add $\int \sin^2(x)dx$ to both sides and divide by 2 to get

$$\int \sin^2(x)dx = \frac{-\sin(x)\cos(x) + x}{2}.$$

23. $\int x\sin(x)\cos(x)dx$

[Hint: let $u = x$. You will need to use the result of the previous problem.]

Let $u = x$ and $dv = \sin(x)\cos(x)dx$. Then $v = \frac{1}{2}\sin^2(x)$, so

$$\begin{aligned}\int x\sin(x)\cos(x)dx &= \frac{x}{2}\sin^2(x) - \frac{1}{2}\int \sin^2(x)dx \\ &= \frac{x}{2}\sin^2(x) - \frac{1}{2}\left[\frac{-\sin(x)\cos(x) + x}{2}\right] + C\end{aligned}$$

Note that we used $\int \sin^2(x)dx$, which we obtained in the previous problem.

24. $\int x\cos(x)dx$

$$= x\sin(x) + \cos(x) + C$$

$$[u = x, \quad dv = \cos(x)dx]$$

25. $\displaystyle\int x^2\cos(x)dx$

You will need to do integration by parts twice. The first time, let $u = x^2$ and $dv = \cos(x)dx$. Then

$$\int x^2\cos(x)dx = x^2\sin(x) - \int \sin(x)\cdot 2x\,dx.$$

Next, let $u = x$ and $dv = \sin(x)dx$. Then

$$\begin{aligned}\int x^2\cos(x)dx &= x^2\sin(x) - 2\int x\sin(x)dx \\ &= x^2\sin(x) - 2\Big(-x\cos(x) + \int\cos(x)dx\Big) \\ &= x^2\sin(x) + 2x\cos(x) - 2\sin(x) + C.\end{aligned}$$

26. $\displaystyle\int e^x\cos(x)dx$

[Hint: Do integration by parts twice. Look carefully at both sides of the resulting equation.]

Let $u = e^x$ and $dv = \cos(x)dx$. Then

$$\int e^x\cos(x)dx = e^x\sin(x) - \int e^x\sin(x)dx.$$

To solve this new integral, use $u = e^x$ and $dv = \sin(x)dx$. Then

$$\begin{aligned}\int e^x\cos(x)dx &= e^x\sin(x) - \Big[-e^x\cos(x) - \int e^x\Big(-\cos(x)\Big)dx\Big] \\ &= e^x\sin(x) + e^x\cos(x) - \int e^x\cos(x)dx\end{aligned}$$

Now add $\displaystyle\int e^x\cos(x)dx$ to both sides of this equation:

$$2\int e^x\cos(x)dx = e^x\sin(x) + e^x\cos(x).$$

Now divide both sides by 2 to obtain the answer:

$$\int e^x\cos(x)dx = \frac{e^x}{2}\Big(\sin(x)+\cos(x)\Big).$$

WARNING: It is important that in the second step you do not choose $u = \sin(x)$ and $dv = e^x$. If you do, you will end up ''undoing'' the first step.

27. $\int \frac{1}{x^2-4}dx$

$$= \int \left(\frac{-1/4}{x+2} + \frac{1/4}{x-2} \right) dx$$

$$= -\frac{1}{4}\ln|x+2| + \frac{1}{4}\ln|x-2| + C$$

28. $\int \frac{x}{x^2-4}dx$

$$= \int \left(\frac{1/2}{x+2} + \frac{1/2}{x-2} \right) dx$$

$$= \frac{1}{2}\ln|x+2| + \frac{1}{2}\ln|x-2| + C$$

29. $\int \frac{1}{x(x+1)}dx$

$$= \int \left(\frac{1}{x} + \frac{-1}{x+1} \right) dx$$

$$= \ln|x| - \ln|x+1| + C$$

30. $\int \frac{1}{x^2(x+1)}dx$

[Use $\frac{A}{x}, \frac{B}{x^2}, \frac{C}{x+1}$.]

$$= \int \left(\frac{-1}{x} + \frac{1}{x^2} + \frac{1}{x+1} \right) dx$$

$$= -\ln|x| - \frac{1}{x} + \ln|x+1| + C$$

31. $\int \frac{x-1}{x^2-16}dx$

$$= \int \left(\frac{5/8}{x+4} + \frac{3/8}{x-4} \right) dx$$

$$= \frac{5}{8} \ln|x+4| + \frac{3}{8} \ln|x-4| + C$$

32. $\int \frac{x+7}{x^2(x+2)}dx$

You should use partial fractions with $\frac{A}{x}, \frac{B}{x^2} \frac{C}{x+2}$. Solving for the A, B, C, you will get

$$A = -\frac{5}{4}, \quad B = \frac{7}{2}, \quad C = \frac{5}{4}.$$

Then the integral will be equal to

$$\int \left(\frac{-\frac{5}{4}}{x} + \frac{\frac{7}{2}}{x^2} + \frac{\frac{5}{4}}{x+1} \right) dx$$

$$= -\frac{5}{4} \ln|x| + \frac{7}{2} \left(-\frac{1}{x} \right) + \frac{5}{4} \ln|x+1| + C$$

33. $\int \frac{1}{x(x^2+1)}dx$

You should use partial fractions with $\frac{A}{x}, \frac{Bx+C}{x^2+1}$. Solving for A, B, C, you will get

$$A = 1, \quad B = -1, \quad C = 0.$$

Then the integral will be equal to

$$\int \left(\frac{1}{x} + \frac{(-1)x+0}{x^2+1} \right) dx = \int \left(\frac{1}{x} - \frac{x}{x^2+1} \right) dx$$

$$= \ln|x| - \frac{1}{2} \ln|x^2+1| + C$$

Note: To do the second integral, use substitution with $u = x^2+1$. We don't need absolute value signs on the second term, because x^2+1 is always positive.

34. $\int \frac{1}{e^x+1}dx$

[Hint: This one is really tricky. Multiply by $\frac{e^x}{e^x}$ and then use substitution.]

Following the hint you get:

$$\int \frac{1}{e^x+1}dx = \int \frac{e^x}{e^x(e^x+1)}dx = \int \frac{1}{e^x(e^x+1)} \cdot e^x \, dx.$$

Now let $u = e^x$, so that $du = e^x dx$. Substitute this into the integral and you get:

$$\begin{aligned} \int \frac{1}{e^x(e^x+1)} \cdot e^x \, dx &= \int \frac{1}{u(u+1)} \, du \\ &= \int \left(\frac{1}{u} + \frac{-1}{u+1}\right) du \\ &= \ln|u| - \ln|u+1| + C \end{aligned}$$

Put back the x's and you get the answer:

$$\int \frac{1}{e^x+1}dx = \ln|e^x| - \ln|e^x+1| + C.$$

Note: Since e^x is always positive, you don't need the absolute value signs.

35. $\int \frac{1}{1+x^2}\,dx$

$$= \arctan(x) + C$$

$$[x = \tan(\theta) \implies \theta = \arctan(x)]$$

36. $\int \frac{1}{\sqrt{4-x^2}}\,dx$

$$= \arcsin(\frac{x}{2}) + C$$

$$[x = 2\sin(\theta) \implies \theta = \arcsin(\frac{x}{2})]$$

37. $\int \frac{1}{x^2+4x+5}\,dx$

$$= \int \frac{1}{(x+2)^2+1}\,dx = \arctan(x+2) + C$$

$$[x+2 = \tan(\theta) \implies \theta = \arctan(x+2)]$$

38. $\int \frac{x^2}{1+x^2}\,dx$

[Hint: This one is tricky. You might have to add and subtract 1 from the numerator.]

$$= \int \frac{x^2+(1-1)}{x^2+1}\,dx = \int \left(\frac{x^2+1}{x^2+1} - \frac{1}{x^2+1}\right)\,dx$$

$$= \int \left(1 - \frac{1}{x^2+1}\right)dx = x - \arctan(x) + C$$

39. $\displaystyle\int_0^1 \frac{1}{x^2}\,dx$

$$\int_r^1 \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_r^1 = -1+\frac{1}{r} \longrightarrow \infty \ \ (\text{as } \ r\to 0)$$

40. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$

$$\int_1^R \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^R = -\frac{1}{R}+1 \longrightarrow 0+1=1 \ \ (\text{as } \ R\to \infty)$$

41. $\displaystyle\int_e^\infty \frac{\ln x}{x}\,dx$

$$\int_e^R \frac{\ln x}{x}\,dx = \frac{(\ln x)^2}{2}\Big|_e^R = \frac{(\ln R)^2}{2}-\frac{1}{2} \longrightarrow \infty \ \ (\text{as } \ R\to \infty)$$

[Use substitution with $u = \ln x$]

42. $\displaystyle\int_e^\infty \frac{1}{x(\ln x)^2}\,dx$

$$\int_e^R \frac{1}{x(\ln x)^2}\,dx = -\frac{1}{\ln x}\Big|_e^R = -\frac{1}{\ln R}+1 \longrightarrow 0+1=1 \ \ (\text{as } \ R\to \infty)$$

[Use substitution with $u = \ln x$]

43. $\displaystyle\int_0^{\infty} xe^{-x}\,dx$

Use integration by parts with $u = x$ and $dv = e^{-x}dx$. Then

$$\int_0^R xe^{-x}\,dx = \left(-xe^{-x} + \int e^{-x}dx\right)\Big|_0^R = \left(-xe^{-x} - e^{-x}\right)\Big|_0^R$$

$$= \left(-Re^{-R} - e^{-R}\right) - (0 - 1) \longrightarrow 0 + 1 = 1$$

as $R \to \infty$.

Remark:

$$-Re^{-R} - e^{-R} = -\frac{R}{e^R} - \frac{1}{e^R} \longrightarrow 0 \ \ (\text{as} \ \ R \to \infty).$$

44. $\displaystyle\int_0^{\infty} xe^{-x^2}\,dx$

Use substitution with $u = x^2$ and $du = 2x\,dx$. Then

$$\int_0^R xe^{-x^2}\,dx = -\frac{1}{2}e^{-x^2}\Big|_0^R = \left(-\frac{1}{2}e^{-R^2}\right) - \left(-\frac{1}{2}\right) \longrightarrow 0 + \frac{1}{2} = \frac{1}{2}$$

as $R \to \infty$.

Remark:

$$-\frac{1}{2}e^{-R^2} = -\frac{1}{2}\frac{1}{e^{R^2}} \longrightarrow 0 \ \ (\text{as} \ \ R \to \infty).$$

45. $\displaystyle\int_1^{\infty} \frac{\arctan x}{x^2+1}\,dx$

[Hint: What is the derivative of arctan(x)?]

Use substitution with $u = \arctan(x)$ and $du = \dfrac{1}{x^2+1}\,dx$. Then

$$\int_1^R \frac{\arctan x}{x^2+1}\,dx = \frac{(\arctan x)^2}{2}\bigg|_0^R = \frac{1}{2}(\arctan R)^2 - \frac{1}{2}(\arctan 1)^2$$

$$\longrightarrow \frac{1}{2}\left(\frac{\pi}{2}\right)^2 - \frac{1}{2}\left(\frac{\pi}{4}\right)^2 = \frac{3}{32}\pi^2$$

as $R \to \infty$.

Remarks:

1. $\arctan(1) = \dfrac{\pi}{4}$, because

$$\tan\left(\frac{\pi}{4}\right) = \frac{\sin\left(\frac{\pi}{4}\right)}{\cos\left(\frac{\pi}{4}\right)} = \frac{\sqrt{2}/2}{\sqrt{2}/2} = 1.$$

2. $\displaystyle\lim_{R\to\infty} \arctan R = \frac{\pi}{2}$, because

$$\tan\left(\frac{\pi}{2}\right) = \frac{\sin\left(\frac{\pi}{2}\right)}{\cos\left(\frac{\pi}{2}\right)} = \frac{1}{0} = \infty.$$

This should be thought of as a limit, not actual equality.

46. $\displaystyle\int_0^{\infty} f(x)\,dx$, where $f(x) = \begin{cases} \frac{1}{\sqrt{x}} & \text{if } x \le 1 \\ \frac{1}{x^2} & \text{if } x \ge 1 \end{cases}$

$$= \int_0^1 \frac{1}{\sqrt{x}}\,dx + \int_1^{\infty} \frac{1}{x^2}\,dx = \left(2\sqrt{x}\right)\bigg|_0^1 + \left(-\frac{1}{x}\right)\bigg|_1^{\infty} = 2 + 1 = 3$$