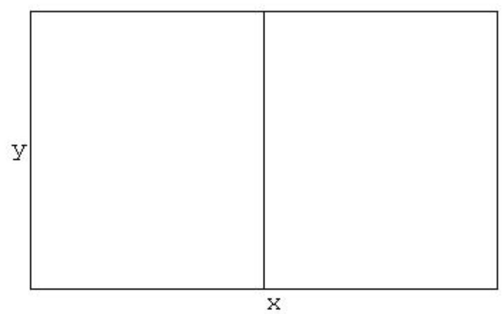
Example (4.7.9) A farmer wants to fence an area of 1.5 million square feet in a rectangle field and then divide it in half with a fence parallel to one of the sides of the rectangle. How can he do this so as to minimize the cost of the fence?

This is an optimization problem where we want to optimize the amount of fence used.

We begin with a diagram:



The width of the fenced region is x, and the length is y.

The total amount of fence used is given by $P = 2x + 3y$. This is what we want to minimize, but we have two variables and we only know how to minimize a function of one variable. We need to introduce another condition to eliminate one of the variables.

The area of the fenced region is $A = 1.5 \times 10^6 = xy$. We can therefore write $x = 1.5 \times 10^6/y$ and then express the amount of fence used as

$$P(y) = \frac{3 \times 10^6}{y} + 3y.$$

The function $P(y)$ is made up of a linear part added to a recipricol part. This means that the function $P(y)$ will have a minimum value for some positive value of y. This minimum will occur when $P'(y) = 0$.

$$P'(y) = -\frac{3 \times 10^6}{y^2} + 3$$

Setting this equal to zero and solving for y, we find $y = \pm 1000$. We can exclude the value less than zero, since y must be greater than zero (it is a length). So we have an extrema at $y = 1000$. We argued above based on the shape of $P(y)$ that this must be a minimum, but we could also verify that this is a minimum by showing $P''(1000) > 0$.

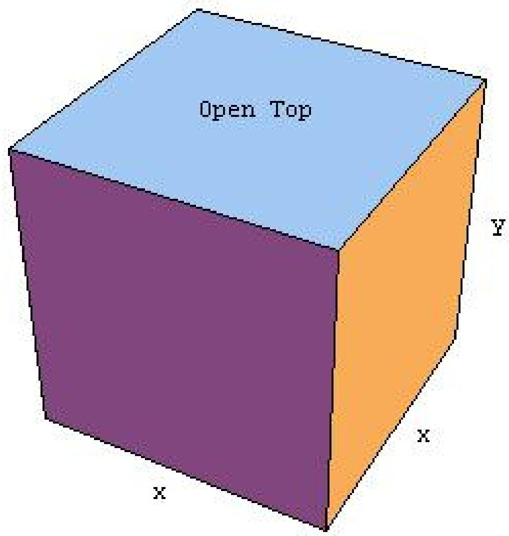
$$P''(y) = \frac{6 \times 10^6}{y^3}$$

Since $P''(y) > 0$ for all $y > 0$, $P(y)$ is always concave up for $y > 0$, and we have found a minimum.

The farmer should build a fence that has three parallel side of length 1000 feet, and the remaining two parallel sides should be of length 1500 feet.

Example (4.7.10) A box with a square base and open top must have a volume of 32,000 cubic centimeters. Find the dimensions of the box that minimize the amount of material used.

Diagram:



The volume is $V = x^2y = 32000$.
The surface area is $S = 4xy + x^2$.
We want to minimize the surface area.
Use the volume relation to get surface area as a function of one variable:
$V = x^2y = 32000 \longrightarrow y = \dfrac{32000}{x^2}$.
$S(x) = 4x\left(\dfrac{32000}{x^2}\right) + x^2 = \dfrac{128000}{x} + x^2$.

Extrema occur when the first derivative is equal to zero, so we need to solve $S'(x) = 0$ for x.

$$\begin{aligned} S'(x) &= -\frac{128000}{x^2} + 2x = 0, \\ \frac{128000}{x^2} &= 2x, \\ 64000 &= x^3, \\ 40 &= x, \end{aligned}$$

We need to check that this is a minimum. We can do that using the second derivative test.

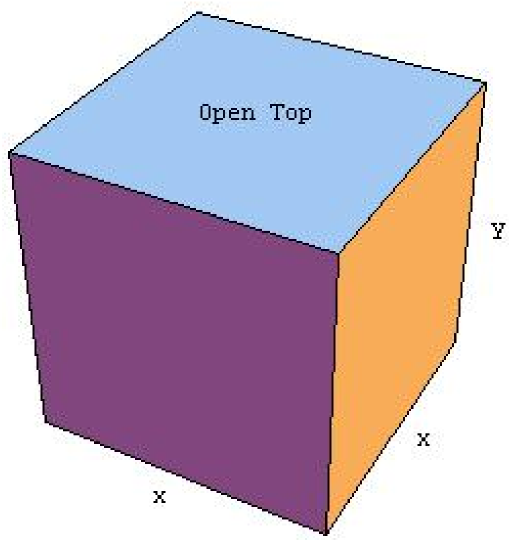
$$S''(x) = \frac{256000}{x^3} + 2,$$

Since $S'(40) = 0$ and $S''(40) > 0$ we know the function $S(x)$ has a minimum at $x = 40$.

Therefore, a box with no top and square base with a volume of 32,000 cubic centimeters has a minimum surface area if the base is a square of side 40 cm and the height is 20 cm.

Example (4.7.11) If 1200 cm^2 of material is available to make a box with a square base and an open top, find the largest possible volume of the box.

Diagram:



The volume is $V = x^2 y$.
The surface area is $S = 4xy + x^2 = 1200$.
We want to maximize the volume.
Use the surface area relation to get volume as a function of one variable:
$S = 4xy + x^2 = 1200 \longrightarrow y = \dfrac{1200 - x^2}{4x}$.
$V(x) = x^2 \left(\dfrac{1200 - x^2}{4x} \right) = \dfrac{1}{4}(1200x - x^3)$.

From the geometry of the situation, $x > 0$ since the volume is zero if $x = 0$.
Also, $x < \sqrt{1200}$ since the volume is also zero there. If we find a critical point inside this interval ($x \in [0, \sqrt{1200}]$), then we will have found a maximum since the volume is zero at the endpoints of the interval.

Another way you could show that you will find a minimum is to use the second derivative test.

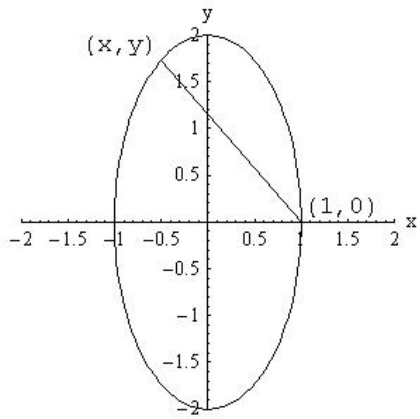
To find the extrema, solve $V'(x) = 0$ for x:
$V'(x) = \frac{1}{4}(1200 - 3x^2) = 0$.

$$\begin{aligned} 1200 - 3x^2 &= 0 \\ x &= \sqrt{\frac{1200}{3}} = \sqrt{400} = 20 \end{aligned}$$

Therefore, a box with no top and square base made from 1200 cm^2 of material will have a maximum volume of $\frac{1}{4}(1200(20) - (20)^3) = 4000$ cm^3 when the base is a square of length 20 cm and the height is 10 cm.

Example (4.7.17) Find the points on the ellipse $4x^2 + y^2 = 4$ that are farthest from the point $(1, 0)$.

Here is a sketch of the ellipse, along with the points of interest.



The point (x, y) is somewhere along the ellipse. From the geometry, we can see that two points will be farthest from $(1, 0)$, one point in the second quadrant, one in the third. They will both have the same x value.

The distance between the point $(1, 0)$ and (x, y) is given by: $d = \sqrt{(1-x)^2 + (0-y)^2}$.
If we minimize the distance squared, we will also minimize the distance. Therefore, work with
$Q = d^2 = (1-x)^2 + y^2$.

This is a function of two variables, so we need to eliminate one, since we only know how to minimize a function of one variable. We can use the equation of the ellipse, $4x^2 + y^2 = 4 \longrightarrow y^2 = 4 - 4x^2$, to help.
$Q(x) = (1-x)^2 + (4 - 4x^2) = 5 - 2x - 3x^2$.

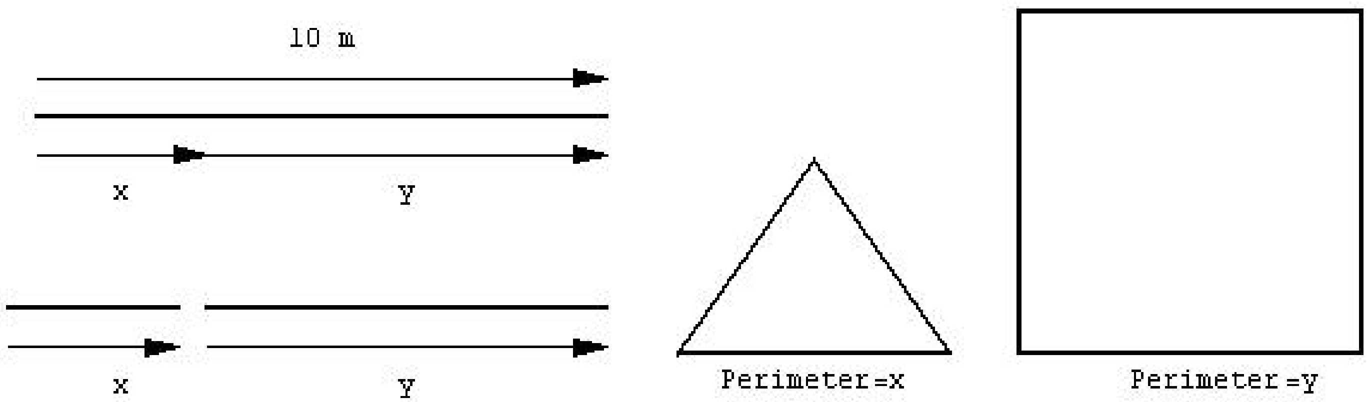
$$\begin{aligned} Q(x) &= 5 - 2x - 3x^2 \\ Q'(x) &= -2 - 6x = 0 \\ -6x &= 2 \\ x &= -\frac{1}{3} \end{aligned}$$

We know this will produce a maximum based on the geometry of the situation.

The two points which are furthest away from $(1, 0)$ are $(1/3, \sqrt{32/9})$ and $(1/3, -\sqrt{32/9})$.

Example (4.7.31) A piece of wire 10 m long is cut into two pieces. One piece is bent into a square and the other is bent into an equilateral triangle. How should the wire be cut so that the total area is (a) a maximum? (b) a minimum?

First, some diagrams:



The length of wire is split into two pieces, one of length x, the other of length y.

We must have $x + y = 10$.

The length of wire x is bent into an equalateral triangle, the length of wire y is bent into a square.

We wish to minimize the area enclosed by the square and rectangle.

Area triangle $= \frac{\sqrt{3}}{4}\left(\frac{x}{3}\right)^2 = \frac{\sqrt{3}}{36}x^2$.

Area of square $= \left(\frac{y}{4}\right)^2 = \frac{y^2}{16}$.

Area we want to minimize is $A = \frac{\sqrt{3}}{36}x^2 + \frac{y^2}{16}$.

We need to eliminate x or y from this equation, since we only know how to optimize a function of one variable. Since $x + y = 10$, we have $x = 10 - y$, and we can write

$$A(y) = \frac{\sqrt{3}}{36}(10-y)^2 + \frac{y^2}{16}.$$

Now, solve $A'(y) = 0$ for y:

$$\begin{aligned} A'(y) &= \frac{\sqrt{3}}{36}(2)(10-y)(-1) + \frac{y}{8} \\ 0 &= -\frac{\sqrt{3}}{18}(10-y) + \frac{y}{8} \end{aligned}$$

$$\begin{aligned}
0 &= -\frac{10\sqrt{3}}{18} + \left(\frac{\sqrt{3}}{18} + \frac{1}{8}\right) y \\
0 &= -\frac{10\sqrt{3}}{18} + \left(\frac{8\sqrt{3} + 18}{18 \cdot 8}\right) y \\
y &= \frac{5\sqrt{3}}{9} \cdot \left(\frac{18 \cdot 8}{8\sqrt{3} + 18}\right) \\
&= \frac{40\sqrt{3}}{4\sqrt{3} + 9}
\end{aligned}$$

The easiest way to show that this value of y minimizes the area is to compute the second derivative:

$$A''(y) = \frac{\sqrt{3}}{18} + \frac{1}{8} > 0$$

Since the second derivative is positive everywhere, the function is always concave up. Therefore, we have found the absolute minimum of the area.

The wire should be cut so that a length of $\frac{40\sqrt{3}}{4\sqrt{3} + 9}$ m forms the square.

Since the area function $A(y)$ is always concave up, the maximum area will occur at the endpoint, when we use the entire length of wire to create either the triangle or the square.

$A(0) = \frac{\sqrt{3}}{36}(10 - 0)^2 + \frac{0^2}{16} = \frac{10\sqrt{3}}{36}$ (triangle only)

$A(10) = \frac{\sqrt{3}}{36}(10 - 10)^2 + \frac{10^2}{16} = \frac{100}{16}$ (square only)

The larger of these two numbers is the second, and the maximum area is produced when the entire wire is used to produce the square.