Examples: Nullspace, Row Space & Column Space

Example 1: Given the 3X4 matrix A = $\begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$ Find:

a) The solution(s) to the system $A\vec{x} = \vec{0}$
b) A basis for the nullspace of A, and its dimension.
c) A basis for the row space of A, and its dimension.
d) A basis for the column space of A, and its dimension.

Solution:

a) The solution(s) to the system $A\vec{x} = \vec{0}$

The augmented matrix for the system, $\begin{bmatrix} 1 & 4 & 5 & 2 & 0 \\ 2 & 1 & 3 & 0 & 0 \\ -1 & 3 & 2 & 2 & 0 \end{bmatrix}$ can be row reduced to

$\begin{bmatrix} 1 & 4 & 5 & 2 & 0 \\ 0 & 1 & 1 & \frac{4}{7} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ which yields the solution $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -s - \frac{2}{7}t \\ -s - \frac{4}{7}t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -\frac{2}{7} \\ -\frac{4}{7} \\ 0 \\ 1 \end{bmatrix}$.

Notice there are two parameters, s & t. x_3 and x_4 are called free variables, x_1 and x_2 are called leading variables. So in this example there are two free variables and two leading variables.

b) A basis for the nullspace of A, and its dimension.

Using part (a), a basis for the nullspace of A is $\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -\frac{2}{7} \\ -\frac{4}{7} \\ 0 \\ 1 \end{bmatrix}$.

Since there are two vectors in this basis, the dimension of the nullspace of A is 2. This is called the <u>nullity</u>. **Nullity(A)=2.**

c) A basis for the row space of A, and its dimension.

Row reducing… $\begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & 1 & 1 & \frac{4}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix}$. So a basis is $\begin{bmatrix} 1 & 4 & 5 & 2 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 & 1 & \frac{4}{7} \end{bmatrix}$ and the **dimension of the row space of A is 2**.

d) A basis for the column space of A, and its dimension.
Two ways you can do this.

1) Using the row echelon form of A in part (c) we can see that the first two columns form a basis for the column space of *that* matrix, thus the first two columns of A form a basis for the column space of A. So a basis is $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$ and the **dimension of the column space of A is 2**
2) Since the column space of A = the row space of A^T we can find a basis for the row space of A^T.

$$A^T = \begin{bmatrix} 1 & 2 & -1 \\ 4 & 1 & 3 \\ 5 & 3 & 2 \\ 2 & 0 & 2 \end{bmatrix} \rightarrow \cdots \rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$. Now since $\begin{matrix} [1 & 2 & -1] \\ [0 & 1 & -1] \end{matrix}$ is a basis for the row space of A^T $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ is a basis for the column space of A.

Example 2: Given the 2X2 matrix A = $\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$ Find:

a) The solution(s) to the system $A\vec{x} = \vec{0}$
b) A basis for the nullspace of A, and its dimension.
c) A basis for the row space of A, and its dimension.
d) A basis for the column space of A, and its dimension.

Solution:

a) The solution(s) to the system $A\vec{x} = \vec{0}$

The augmented matrix for the system, $\begin{bmatrix} 1 & 3 & 0 \\ 2 & 5 & 0 \end{bmatrix}$ can be row reduced to $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ which yields the solution $\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. We should have expected this since the det(A) ≠ 0, so A is invertible and $A\vec{x} = \vec{0}$ has only the trivial solution.

b) A basis for the nullspace of A, and its dimension.

Using part (a), the nullspace is just the zero space so there is no basis and the dimension is zero. **Nullity(A)=0.**

c) A basis for the row space of A, and its dimension.

Row reducing… $\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$. So a basis is $\begin{matrix} [1 & 3] \\ [0 & 1] \end{matrix}$ and the **dimension of the row space of A is 2**. Notice, in this case, since it is easy to see that the rows of A are two linearly independent vectors, the row space is R^2 so any two linearly independent vectors will form a basis.

d) A basis for the column space of A, and its dimension.

Using the row echelon form of A in part (c) we can see that the two columns form a basis for the column space of that matrix, thus the two columns of A form a basis for the column space of A. So a basis is $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ and the **dimension of the column space of A is 2**

Example 3:

(a) Find a basis for the row space of A= $\begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$ that consists entirely of row vectors of A.

(b) Find a subset of the vectors $\mathbf{v}_1$=(1, 4, 5, 2), $\mathbf{v}_2$= (2, 1, 3, 0), $\mathbf{v}_3$= (-1, 3, 2, 2) that forms a basis for the space spanned by these three vectors.

(c) Express each vector not in the basis as a linear combination of the basis vectors.

Solution:
(a) Since row space of A = the column space of A^T, we'll find the column space of A using the first method in example 1d.

$$A^T = \begin{bmatrix} 1 & 2 & -1 \\ 4 & 1 & 3 \\ 5 & 3 & 2 \\ 2 & 0 & 2 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Using the row echelon form of A^T we can see that the first two columns form a basis for the column space of *that* matrix, thus the first two columns of A^T form a basis for the column space of A^T . Thus $\begin{bmatrix} 1 & 4 & 5 & 2 \end{bmatrix}$ $\begin{bmatrix} 2 & 1 & 3 & 0 \end{bmatrix}$ form a basis for the row space of A.

(b) This is the same problem, other than notation. So $\mathbf{v}_1$=(1, 4, 5, 2) & $\mathbf{v}_2$= (2, 1, 3, 0) form a basis for the space spanned by the three given vectors.

(c) Since $\mathbf{v}_1$ & $\mathbf{v}_2$ form a basis for the space spanned by the three given vectors, $\mathbf{v}_3$ can be written as a linear combination of $\mathbf{v}_1$ & $\mathbf{v}_2$. This combination is most easily found by continuing the row operations from part (a) to obtain *reduced* row echelon form.

$$A^T = \begin{bmatrix} 1 & 2 & -1 \\ 4 & 1 & 3 \\ 5 & 3 & 2 \\ 2 & 0 & 2 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} .$$

From here we can see that the third column is one times the first column – one times the second which holds true in A^T as well. Since the columns of A^T are precisely the given vectors, we have **$\mathbf{v}_3 = \mathbf{v}_1 - \mathbf{v}_2$**