Illustration of why the [A|I] method of finding A^{-1} works.
(College Algebra, 3rd edition, Mark Dugopolski)

If we are given the matrix

$$A = \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}$$

from Example 2, how do we find its inverse if it is not already known? According the definition, A^{-1} is a 2×2 matrix such that $AA^{-1} = I$ and $A^{-1}A = I$. So if

$$A^{-1} = \begin{bmatrix} x & y \\ z & w \end{bmatrix},$$

then

$$\begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x & y \\ z & w \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

To find A^{-1} we solve these matrix equations. If A is invertible, both equations have the same solution. We will work with the first one. Multiply the two matri on the left-hand side of the first equation to get the following equation:

$$\begin{bmatrix} 3x + 4z & 3y + 4w \\ 5x + 7z & 5y + 7w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Equate the corresponding terms from these equal matrices to get the following systems:

$$3x + 4z = 1 \qquad 3y + 4w = 0$$
$$5x + 7z = 0 \qquad 5y + 7w = 1$$

We can solve these two systems by using the Gaussian elimination method from Section 6.1. The augmented matrices for these systems are

$$\left[\begin{array}{cc|c} 3 & 4 & 1 \\ 5 & 7 & 0 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{cc|c} 3 & 4 & 0 \\ 5 & 7 & 1 \end{array}\right].$$

Note that the two augmented matrices have the same coefficient matrix. Since we would use the same row operations on each of them, we can solve the systems simultaneously by combining the two systems into one augmented matrix denoted $[A|I]$:

$$[A|I] = \left[\begin{array}{cc|cc} 3 & 4 & 1 & 0 \\ 5 & 7 & 0 & 1 \end{array}\right]$$

So the problem of finding A^{-1} is equivalent to the problem of solving two systems by Gaussian elimination. A is invertible if and only if these systems have a solution.