6.3 Notes Gram Schmidt Orthonormalization Process

Goal: Given an inner product space V and a basis $B=\{\vec{u}_1,\vec{u}_2,\ldots\vec{u}_n\}$, find an orthonormal basis.

Two approaches:

(1) Find an orthogonal basis $B'=\{\vec{v}_1,\vec{v}_2,\ldots\vec{v}_n\}$ and then normalize.

Let $\vec{v}_1=\vec{u}_1$. Now let W_1 be the space spanned by $\vec{v}_1$ (i.e. $W_1=\text{span}(\vec{v}_1)$).

$$\vec{v}_2=\vec{u}_2-proj_{W_1}\vec{u}_2=\vec{u}_2-\frac{\langle\vec{u}_2,\vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1.$$

$$\vec{v}_3=\vec{u}_3-proj_{W_2}\vec{u}_3=\vec{u}_3-\frac{\langle\vec{u}_3,\vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1-\frac{\langle\vec{u}_3,\vec{v}_2\rangle}{\|\vec{v}_2\|^2}\vec{v}_2 \quad ; \quad W_2=\text{span}(\vec{v}_1,\ \vec{v}_2)$$

$$\vec{v}_4=\vec{u}_4-proj_{W_3}\vec{u}_4=\vec{u}_4-\frac{\langle\vec{u}_4,\vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1-\frac{\langle\vec{u}_4,\vec{v}_2\rangle}{\|\vec{v}_2\|^2}\vec{v}_2-\frac{\langle\vec{u}_4,\vec{v}_3\rangle}{\|\vec{v}_3\|^2}\vec{v}_3 \ ; \quad W_3=\text{span}(\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3)$$

$\vdots$

$$\vec{v}_n=\vec{u}_n-proj_{W_{n-1}}\vec{u}_n;\ W_{n-1}=\text{span}(\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3\ldots\ \vec{v}_n)$$

(2) Normalize as you go. (Note, since the growing basis is orthonormal we can use the simpler formula for projection here)

$$\vec{v}_1=\frac{\vec{w}_1}{\|\vec{w}_1\|} \text{ where } \vec{w}_1=\vec{u}_1$$

$$\vec{v}_2=\frac{\vec{w}_2}{\|\vec{w}_2\|} \text{ where } \vec{w}_2=\vec{u}_2-proj_{W_1}\vec{u}_2=\vec{u}_2-\langle\vec{u}_2,\vec{v}_1\rangle\vec{v}_1$$

$$\vec{v}_3=\frac{\vec{w}_3}{\|\vec{w}_3\|} \text{ where } \vec{w}_3=\vec{u}_3-proj_{W_2}\vec{u}_3=\vec{u}_3-\langle\vec{u}_3,\vec{v}_1\rangle\vec{v}_1-\langle\vec{u}_3,\vec{v}_2\rangle\vec{v}_2$$

$\vdots$

$$\vec{v}_n=\frac{\vec{w}_n}{\|\vec{w}_n\|} \text{ where } \vec{w}_n=\vec{u}_n-proj_{W_{n-1}}\vec{u}_n$$

Good Illustration of process:
http://www.khanacademy.org/video/linear-algebra--the-gram-schmidt-process?playlist=Linear%20Algebra

Good Examples:
http://tutorial.math.lamar.edu/Classes/LinAlg/OrthonormalBasis.aspx

http://www.khanacademy.org/video/linear-algebra--gram-schmidt-process-example?playlist=Linear%20Algebra

http://www.khanacademy.org/video/linear-algebra---gram-schmidt-example-with-3-basis-vectors?playlist=Linear%20Algebra