## Infinite Series Introductory Example

Stand 2 meters from the wall. At each turn step half the distance to the wall. Will you reach the wall? If so, what is the total distance traveled?

| Turn Number $n$ | Length of Step $a_n$ | Distance $s_n$ | |
|---|---|---|---|
| 1 | 1 | 1 | =1 |
| 2 | $\frac{1}{2}$ | $1+\frac{1}{2}$ | $=\frac{3}{2}$ |
| 3 | $\frac{1}{4}$ | $1+\frac{1}{2}+\frac{1}{4}$ | $=\frac{7}{4}$ |
| 4 | $\frac{1}{8}$ | $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}$ | $=\frac{15}{8}$ |
| 5 | $\frac{1}{16}$ | $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}$ | $=\frac{31}{16}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | |
| n | $\frac{1}{2^{n-1}}$ | $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots+\frac{1}{2^{n-1}}$ | $=??$ |
| $\vdots$ | $\vdots$ | $\vdots$ | |

So Total Distance (if we reach wall) would be $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$

Definition: An infinite series is an expression of the form $a_1+a_2+a_3+\cdots$ or $\sum_{n=1}^{\infty} a_n$. We wish to determine if the series has a "sum".
Let $S_n$ be the sum of the first n terms ( $S_1=a_1$, $S_2=a_1+a_2$ $S_3=a_1+a_2+a_3, \ldots$ $S_n=a_1+a_2+a_3+..+a_n$). $S_n$ is called the nth partial sum.
Now form the sequence of partial sums: $S_1$, $S_2$, $S_3, \ldots$ . If this sequence of partial sum converges to a finite limit S (i.e. $\lim_{n\to\infty} s_n = s$) then the series is said to converge and the sum is defined to be S, that is $\sum_{n=1}^{\infty} a_n$ = S. Otherwise, the series diverges.

## VOCABULARY/ NOTATION

| General | Above Example |
|---|---|
| Series: $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$ | $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$ |
| Terms of the series: $a_1, a_2\ a_3, \cdots$ | $1,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \frac{1}{16}, \cdots$ |
| nth term of the series: $a_n$ | $\frac{1}{2^{n-1}}$ |
| nth partial sum: $s_n = a_1 + a_2 + a_3 + \cdots + a_n$ | $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^{n-1}}$ |
| sequence of partial sums: $s_1,\ s_2,\ s_3,$ | $1,\ \frac{3}{2},\ \frac{7}{4},\ \frac{15}{8},\ \frac{31}{16}, \cdots,\ ??$ |
| nth term partial sums: $s_n$ in "closed form" | ?? |

To determine convergence of series directly, using the definition:
Construct the sequence of partial sums. Findi the general term of this sequence, $s_n$ (This is usually the hard part. You need to find the pattern or perhaps the series is telescoping or you get otherwise creative.) Then check whether the sequence of partial sums converges by finding $\lim_{n\to\infty} s_n$. If the limit is finite , say $\lim_{n\to\infty} s_n = S$, then the sequence of partial sums converges so the series converges and has sum S. $\left( \sum_{n=1}^{\infty} a_n = S \right)$. If the limit is infinite or does not exist, the sequence of partial sums diverges so the series diverges.